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Conditional Concentration for functions of HD rdm arrays

(joint work with P. Dodos and K. Tyros)

Introduction

- Concentration: (Talagrand) A function of many variables which depends smoothly on them is essentially constant.

Examples
- (Gaussian) $G \sim N(0, I_n)$ and $f: \mathbb{R}^n \to \mathbb{R}$, f is 1-Lipschitz

  then $\mathbb{P}(|f(G) - \mathbb{E}f(G)| > t) \leq 2e^{-t^2/2}$, $t>0$

- Martingale methods: Azuma's inequality

  If $d_1, \dots, d_n$ is a multiplicative system on $(\Omega, \mathcal{E}, P)$, that is

  $$\forall\, I \subset [n] \quad \mathbb{E}\Big[\prod_{i \in I} d_i\Big] = 0,$$

  (e.g. a martingale dif. seq.)

  then $$\Big\| \sum_{j=1}^{n} d_j \Big\|_{L^p} \lesssim \sqrt{p} \Big(\sum_{j=1}^{n} \|d_j\|_{L^\infty}^2 \Big)^{1/2}, \quad 2 \leq p < \infty.$$

- Doob's martingale decomposition + Azuma ineq $\Longrightarrow$

McDiarmid's bd dif. ineq : $f(X_1, \dots, X_n)$, X_i indep.

f is Lip wrt the weighted Hamming metric

$$d(x,y) = \sum_{i=1}^{n} c_i \mathbb{1}_{\{x_i \neq y_i\}}$$

$$f(X) - \mathbb{E} f(X) = \sum_{j=1}^{n} d_j$$

$$d_k = \mathbb{E}[f(X) \mid X_1, \dots, X_k] - \mathbb{E}[f(X) \mid X_1, \dots, X_{k-1}]$$

$$k = 1, 2, \dots, n$$

Q: How about concentration when f lacks smoothness properties ?

Tao $(p=2)$
DKT $(1 < p \leq 2)$ } Conditional concentration

$p=2$: Given $k \in \mathbb{N}$, $\varepsilon > 0$ $\exists$ $N = N(\varepsilon, k) \in \mathbb{N}$ st: For any rdm vector $X = (X_1, \dots, X_n)$ where X_i are indep and $\mathcal{X}$-valued, and for any $f : \mathcal{X}^n \to \mathbb{R}$ with $\|f(X)\|_{L^2} = 1$, there exists a segment $I \subset [n]$ with $|I| = k$ so that $\mathbb{E}[f(X) \mid \underbrace{X_i : i \in I}_{F_I}]$ is well-concentrated around its mean, i.e.

$$\mathbb{P}\left(\left| \mathbb{E}[f(X) \mid F_I] - \mathbb{E}f(X) \right| > \varepsilon \right) \leq \varepsilon .$$

I: ~~interval~~ segment

$I = \{i+1, i+2, \dots, i+k\} \subset [n]$.

<u>Sketch of proof</u>

$$f(X) - \mathbb{E}f(X) = \sum_{k=1}^{n} d_k , \qquad d_k = \text{as before}$$

d_k are mutually orthogonal

$$1 \geq \operatorname{Var}(f(X)) = \mathbb{E}\left[\sum d_i\right]^2 = \sum_{i=1}^{n} \mathbb{E}(d_i^2) \implies \exists\, k \text{ st } \mathbb{E}\, d_k^2 \leq \frac{1}{n}$$

$$\frac{1}{n} \geq \mathbb{E}\left(\mathbb{E}[f \mid X_1 \dots X_k] - \mathbb{E}[f \mid X_1 \dots X_{k-1}] \right)^2$$

$$= \mathbb{E}\left[\mathbb{E}[d_k^2 \mid X_k] \right] \underset{\text{Jensen}}{\geq} \mathbb{E}\left[\left(\mathbb{E}[d_k \mid X_k] \right)^2 \right]$$

$\uparrow$ X_i's indeep.

$$\mathbb{E}\left(\mathbb{E}[f \mid X_k] - \mathbb{E}[f] \right)^2$$

If instead $I_1, \dots, I_m$, $m \sim \frac{n}{k}$, $|I_j| = k$ intervals of $[n]$

and $\Delta_k = \mathbb{E}\left[f \mid X_i : i \in \bigcup_{s=1}^{t} I_s\right] - \mathbb{E}\left[f \mid X_i : i \in \bigcup_{s=1}^{k-1} I_s\right]$.

by arguing similarly, we find that $\exists\, k \in \{1, 2, \dots, m\}$ s.t.

$$\mathrm{Var}\left(\mathbb{E}[f \mid X_i : i \in I_k] \right) \leq \frac{1}{m} \sim \frac{k}{n}.$$

$$\mathbb{P}\left(\left| \mathbb{E}[f \mid X_i : i \in I_k] - \mathbb{E} f(X) \right| > \varepsilon \right) \lesssim \frac{1}{\varepsilon^2} \frac{k}{n} \leq \varepsilon,$$

provided that $k \lesssim \varepsilon^3 n$.

<u>DKT (2016)</u>: Similar for $\|f(X)\|_{L^p} = 1$, $1 < p \leq 2$.

Extension of the "orthog. of m.d.s" in L^p:

- Burkholder's ineq: estimates for square-function,
- Ricard, Xu (2016): $\frac{1}{\sqrt{p-1}} \left\| \sum_{i=1}^m d_i \right\|_{L^p} \geq \left(\sum_{i=1}^m \|d_i\|_{L^p}^2 \right)^{1/2}$. (d_i) m.d.s.

<u>Main Goal:</u> To extend this phenomenon to HD rdm arrays, that lack independence, but enjoy symmetries.

Def. (rdm arrays ; subarrays)

- d-dim rdm array on $[n]$ it's a stochastic process $X = \langle X_t : t \in \binom{[n]}{d} \rangle$
- sub-array : fix $I \subset [n]$, $|I| \geq d$: $X_I = \langle X_t : t \in \binom{I}{d} \rangle$
- σ-algebras generated by X_I will be denoted by $\mathcal{F}_I = \sigma(X_I)$

Examples 1) $d = 1 \rightsquigarrow$ rdm vectors.

$d = 2 \rightarrow$ rdm sym. matrices

$d \gg 1 \rightsquigarrow$ rdm sym. tensors.

2) $\xi_1, \dots, \xi_n$ (indep) rdm variables $\forall\, t \subset [n]$, $|t| = d \Longrightarrow X_t = \prod_{j \in t} \xi_j$.

or more generally $X_t = f(\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_d})$,

when $t = \{ i_1 < i_2 < \dots < i_d \} \subset [n]$

- spreadability: X d-dim rdm array it's spreadable if

$\forall\, A, B \subset [n]$, $|A| = |B| \geq d$ $\quad X_A \overset{d}{=} X_B$.

η-spreadable : —— // —— $\quad d_{TV}(\mathcal{L}(X_A), \mathcal{L}(X_B)) \leq \eta$.

FACT If a rdm array is finite-valued then for n large enough there is a sub-array which is η-spreadable. (Hence, approx. spreadability is ubiquitous)

dissociativity: X dissociated if $\forall A, B \subset [n]$, $A \cap B = \emptyset \Rightarrow X_A, X_B$ are indep.

E.g. $X_t = \prod_{i \in t} \xi_i$, (ξ_i) indep. (as above)

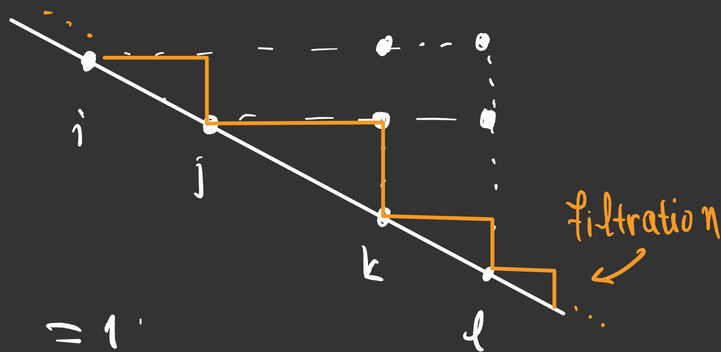
Main result (for $d=2$; Boolean case)

$1 < p \leq 2$, $\varepsilon > 0$, $k \geq 2$ and $C := C(\varepsilon, p, k) = \exp\left(\frac{c_0}{\varepsilon^8 (p-1)} k^2\right)$

$n \geq C$ and $X = \langle X_t : t \in \binom{[n]}{2} \rangle$ be $\{0,1\}$-valued,

$\frac{1}{C}$-spreadable, and assume the

"Box independence condition, i.e.,



$$\left| \mathbb{E}[X_{13} X_{14} X_{23} X_{24}] - \mathbb{E}[X_{13}]\mathbb{E}[X_{14}]\mathbb{E}[X_{23}]\mathbb{E}[X_{24}] \right| \leq \frac{1}{C}.$$

Then, $\forall f : \{0,1\}^{\binom{[n]}{2}} \to \mathbb{R}$ s.t. $\| f(X) \|_{L^p} = 1$,

$\exists I \subset [n]$, $|I| = k \left(\simeq_{\varepsilon, p} \sqrt{\log n} \right)$ s.t. $\mathbb{P}\left(\left| \mathbb{E}[f(X) \mid \mathcal{F}_I] - \mathbb{E} f(X) \right| > \varepsilon \right) < \varepsilon$.