On Multiple L_p-curvilinear-Brunn-Minkowski inequality

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Section 1

Minkowski summation

α-mean of numbers

For $a, b \geq 0$, $\alpha \in [-\infty, \infty]$ and $t \in [0, 1]$,

$$M_\alpha^t(a,b) = \begin{cases} [(1-t)a^\alpha + tb^\alpha]^{\frac{1}{\alpha}}, & \text{if } \alpha \neq 0, \pm\infty, \\ a^{1-t}b^t, & \text{if } \alpha = 0, \\ \max\{a,b\}, & \text{if } \alpha = +\infty, \\ \min\{a,b\}, & \text{if } \alpha = -\infty, \end{cases}$$

if $ab > 0$, and $M_\alpha^t(a,b) = 0$ if $ab = 0$.

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For $A, B \subset \mathbb{R}^n$,

$$\begin{aligned} &(1-t)A + tB \\ = \ &\{(1-t)x + ty : x \in A, y \in B\} \\ = \ &\{\big(M_1^t(x_1,y_1), \cdots, M_1^t(x_n,y_n)\big) : x = (x_1, \cdots, x_n) \in A, y = (y_1, \cdots, y_n) \in B\}. \end{aligned}$$

Brunn-Minkowski inequality

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For any measurable sets $A, B \subset \mathbb{R}^n$,

$$V_n((1-t)A + tB) \geq M^t_{1/n}(V_n(A), V_n(B)),$$

with equality holds if and only if A and B are homothetic, where $V_n(\cdot)$ denotes the volume (Lebesgue measure) for sets in $\mathbb{R}^n$.

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Curvilinear convex combination (Uhrin, ADV, 1994)

For $A, B \subset \mathbb{R}^n \times \mathbb{R}_+$, $t \in (0,1)$, and $\alpha \in [-\infty, \infty]$,

$$\begin{aligned} &(1-t) \times_\alpha A +_\alpha t \times_\alpha B \\ =& \left\{ \left((1-t)x + ty, M^t_\alpha(a,b)\right) : (x,a) \in A, (y,b) \in B \right\} \\ =& \left\{ \left(M^t_1(x_1,y_1), \cdots, M^t_1(x_n,y_n), M^t_\alpha(a,b)\right) : (x_1, \cdots, x_n, a) \in A, (y_1, \cdots, y_n, b) \in B \right\}. \end{aligned}$$

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- $\alpha = 1$: Minkowski summation in $\mathbb{R}^{n+1}$.

Compression

Hypo-graph

For $f\colon \mathbb{R}^n \to \mathbb{R}_+ = [0,\infty)$,

$$\mathrm{hyp}(f) := \{(x,r) \in \mathbb{R}^n \times \mathbb{R}_+ : 0 \le r \le f(x)\}.$$

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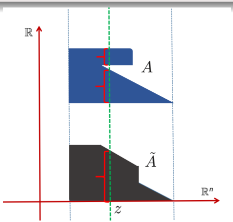
Compression (Blaschke, 1917)

The compression of $A \subset \mathbb{R}^n \times \mathbb{R}_+$ which is also known as shaking, is

$$\tilde{A} = \mathrm{hyp}(V_A),$$

where the segment function $V_A(z) = V_1(A \cap (\mathbb{R}_+ + z)), A \cap (z + \mathbb{R}_+) \neq \emptyset, z \in \mathbb{R}^n$.



Curvilinear Brunn-Minkowski inequality

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For $\alpha \in [-\infty, \infty]$ and any bounded Borel subsets $A, B \subset \mathbb{R}^n \times \mathbb{R}_+$, each having finite positive volume, and $t \in (0,1)$, one has

$$V_{n+1}((1-t) \times_\alpha A +_\alpha t \times_\alpha B) \geq \begin{cases} M^t_{\frac{\alpha}{1+n\alpha}}(V_{n+1}(A), V_{n+1}(B)), & \text{if } \alpha \geq -\frac{1}{n}, \\ \min\left\{(1-t)^{\frac{1+n\alpha}{\alpha}} V_{n+1}(A), t^{\frac{1+n\alpha}{\alpha}} V_{n+1}(B)\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}$$

- $\alpha = 1$: Brunn-Minkowski inequality in $\mathbb{R}^{n+1}$.

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- $\alpha = 1$: Brunn-Minkowski inequality in $\mathbb{R}^{n+1}$.
- (i) $V_n(\tilde{A}) = V_n(A)$, $V_n(\tilde{B}) = V_n(B)$.
  (ii) $V_{n+1}((1-t)\times_\alpha A +_\alpha t\times_\alpha B) \geq V_{n+1}((1-t)\times_\alpha \tilde{A} +_\alpha t\times_\alpha \tilde{B})$.
  (iii)
$$V_{n+1}((1-t)\times_\alpha \tilde{A} +_\alpha t\times_\alpha \tilde{B}) \geq \begin{cases} M^t_{\frac{\alpha}{1+n\alpha}}(V_{n+1}(\tilde{A}), V_{n+1}(\tilde{B})), & \text{if } \alpha \geq -\frac{1}{n}, \\ \min\left\{(1-t)^{\frac{1+n\alpha}{\alpha}} V_{n+1}(\tilde{A}), t^{\frac{1+n\alpha}{\alpha}} V_{n+1}(\tilde{B})\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}$$

Section 2

Borell-Brascamp-Lieb inequality

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$$h((1-t)x + ty) \geq M_\alpha^t(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$. Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \begin{cases} M^t_{\frac{\alpha}{1+n\alpha}} \left(\int_{\mathbb{R}^n} f(x)dx, \int_{\mathbb{R}^n} g(x)dx\right), & \text{if } \alpha \geq -\frac{1}{n}, \\ \min\left\{(1-t)^{\frac{1+n\alpha}{\alpha}} \int_{\mathbb{R}^n} f(x)dx, t^{\frac{1+n\alpha}{\alpha}} \int_{\mathbb{R}^n} g(x)dx\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}$$

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- Methods:
  (1) Mass transportation;
  (2) Hypo-graphs for functions and Curvilinear Brunn-Minkowski inequality, etc.

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  (1) Mass transportation;
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- $\alpha = 0$: Prékopa-Leindler inequality.
- $f = \chi_K$, $g = \chi_L$: Dimension-free Brunn-Minkowski inequality for convex bodies K, L.

Section 3

L_p coefficients (Quasilinearzation)

L_p coefficients: For $t, \lambda \in (0,1)$, $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, denote

$$C_{p,\lambda,t} := (1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}} \text{ and } D_{p,\lambda,t} := t^{\frac{1}{p}}\lambda^{\frac{1}{q}}.$$

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L_p α-mean of numbers

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- $0 < p < 1$: $\inf_{0<\lambda<1}(C_{p,\lambda,t}a + D_{p,\lambda,t}b) = M_p^t(a,b)$.

L_p Brunn-Minkowski inequality

Generalized L_p Minkowski summation (LYZ, AAM, 2012)

For measurable sets $A, B \subset \mathbb{R}^n$, the L_p Minkowski summation is defined as

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- $p = 1$: the classical Brunn-Minkowski inequality for measurable sets.

Multiple L_p-curvilinear summation

$L_{p,\bar{\alpha}}$-curvilinear summation (Roysdon-Xing, 2022)

For $p \geq 1$, $A, B \subset \mathbb{R}_+^{n+1}$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$ where $\alpha_i \in [-\infty, \infty]$ for each $i = 1, \ldots, n+1$, $(x_1, \ldots, x_{n+1}) \in A$, $(y_1, \ldots, y_{n+1}) \in B$,

$$\begin{aligned} &(1-t) \otimes_{p,\bar{\alpha}} A \oplus_{p,\bar{\alpha}} t \otimes_{p,\bar{\alpha}} B \\ := &\bigcup_{0<\lambda<1} \left\{ \left(M_{p,\alpha_1}^{(t,\lambda)}(x_1, y_1), \ldots, M_{p,\alpha_{n+1}}^{(t,\lambda)}(x_{n+1}, y_{n+1}) \right) \right\}, n \geq 0. \end{aligned}$$

For $0 < p < 1$, replace $\bigcup_{0<\lambda<1}$ by $\bigcap_{0<\lambda<1}$ above.

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- ★ $\bar{\alpha} = (1, \ldots, 1, \alpha)$: $L_{p,\alpha}$-curvilinear combination,

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$$= \begin{cases} \bigcup_{0<\lambda<1} \left\{ \left(C_{p,\lambda,t} x + D_{p,\lambda,t} y, M^{(t,\lambda)}_{p,\alpha}(a, b) \right) : (x, a) \in A, (y, b) \in B \right\}, & n \geq 0, \\ \bigcup_{0<\lambda<1} \left\{ M^{(t,\lambda)}_{p,\alpha}(a, b) : a \in A, b \in B \right\}, & n = 0. \end{cases}$$

- ★ $\bar{\alpha} = (1, \ldots, 1, 1)$: $(1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B = (1-t) \cdot_p A +_p t \cdot_p B$.
- ★ $p = 1$: $(1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B = (1-t) \times_{\alpha} A +_{\alpha} t \times_{\alpha} B$.

$L_{p,\bar{\alpha}}$-curvilinear Brunn-Minkowski inequality

$L_{p,\bar{\alpha}}$-curvilinear Brunn-Minkowski inequality

Let $p \geq 1$, $t \in (0,1)$, and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$ with $\alpha_i \in (0,1]$ for $i = 1, \ldots, n$. Suppose that $A, B \subset (\mathbb{R}_+)^{n+1}$ are bounded Borel sets of positive volume, then

$$V_{n+1}((1-t) \otimes_{p,\bar{\alpha}} A \oplus_{p,\bar{\alpha}} t \otimes_{p,\bar{\alpha}} B)$$
$$\geq \begin{cases} M^t_{p\gamma}(V_{n+1}(A), V_{n+1}(B)), & \text{if } \alpha_{n+1} \geq \beta, \\ \sup\left\{\min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(A), [D_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(B)\right\} : 0 < \lambda < 1\right\}, & \text{if } \alpha_{n+1} < \beta, \end{cases}$$

where $\beta = -\left(\sum_{i=1}^n \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_{i=1}^{n+1} \alpha_i^{-1}\right)^{-1}$.

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where $\beta = -\left(\sum_{i=1}^n \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_{i=1}^{n+1} \alpha_i^{-1}\right)^{-1}$.

- $\bar{\alpha} = (1, \cdots, 1)$:
  (i): $(1-t) \otimes_{p,\bar{\alpha}} A \oplus_{p,\bar{\alpha}} t \otimes_{p,\bar{\alpha}} B = (1-t) \cdot_p A +_p t \cdot_p B$: the L_p Minkowski summation.
  (ii): $L_{p,\bar{\alpha}}$-curvilinear-Brunn-Minkowski inequality reduces to the L_p Brunn-Minkowski inequality in $\mathbb{R}^{n+1}$.

Main idea: Mathematical induction

- Step 1: $\dim = 1$. Let $p \geq 1$, $t \in (0,1)$, and $\alpha \in [-\infty, 1]$ and sets $A, B \subset \mathbb{R}_+$ be of the form $A = \bigcup_{i=1}^{m} [a_i, b_i], B = \bigcup_{j=1}^{n} [c_j, d_j]$, where the intervals in K and L have mutually disjoint interiors and $m, n \in \mathbb{N}$. Then $V_1((1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B) \geq M_{p\alpha}^t(V_1(A), V_1(B))$.

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- Step 2: $\dim = n+1$ and Compression. Let $p \geq 1$, $t \in (0,1)$, $\alpha \in [-\infty, 1]$ and $A, B \subset \mathbb{R}^n \times \mathbb{R}_+$ with $V_{n+1}(A), V_{n+1}(B) > 0$. Then

$$V_{n+1}((1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B) \geq V_{n+1}((1-t) \times_{p,\alpha} \tilde{A} +_{p,\alpha} t \times_{p,\alpha} \tilde{B}).$$

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- Step 3: $\dim = n+1$ and $L_{p,\alpha}$-curvilinear-Brunn-Minkowski inequality. Let $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (0,1)$, $\alpha \in [-\infty, \infty]$ and $A, B \subset \mathbb{R}^n \times \mathbb{R}_+$ with $A = \tilde{A}, B = \tilde{B}$. Then for $\gamma = \frac{\alpha}{1+n\alpha}$,

$$V_{n+1}((1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B)$$
$$\geq \begin{cases} M_{p\gamma}^{t}(V_{n+1}(A), V_{n+1}(B)), & \text{if } \alpha \geq -\frac{1}{n}, \\ \sup_{0<\lambda<1} \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(A), [D_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(B)\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}$$

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$$V_{n+1}((1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B) \geq \begin{cases} M_{p\gamma}^t(V_{n+1}(A), V_{n+1}(B)), & \text{if } \alpha \geq -\frac{1}{n}, \\ \sup_{0<\lambda<1} \min\left\{ [C_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(A), [D_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(B) \right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}$$

- Step 4: Repeat the above with respect to the multiple power parameters.

Normalized L_p curvilinear BMI

- Generalized segment function: $V_{A,H}(y) = V_{k+1}(A \cap (\bar{H} + y))$, where $H \subset G_{n,k}$ (Grassmannian manifold), $k \in \{0, 1, \cdots, n\}$, $y \in H^\perp$ and $\bar{H} = H \times \mathbb{R}_+$ for $A \subset \mathbb{R}^n \times \mathbb{R}_+$.
  $k = 0$: the classic segment function.

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  $k = 0$: the classic segment function.
- Optimal norm: $\|V_{A,H}\|_\infty = \sup_{y \in H^\perp} V_{A,H}(y)$.
- Normalized super-level set: $C_r(V_{A,H}) = \left\{y \in H^\perp \colon V_{A,H}(y) \geq r\|V_{A,H}\|_\infty\right\}$ for $0 \leq r \leq 1$.

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- Normalized compression:
  $$A_H = \left\{ (y, r) \in H^\perp \times \mathbb{R}_+ : 0 \leq r \leq \frac{V_{A,H}(y)}{\|V_{A,H}\|_\infty},\ A \cap (\bar{H} + y) \neq \emptyset \right\}.$$

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Normalized version

Let $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (0, 1)$, $A, B \subset \mathbb{R}^n \times \mathbb{R}_+$, $A = \tilde{A}$, $B = \tilde{B}$. Then

$$V_{n+1}((1-t) \times_{p,\alpha} A +_{p,\alpha} t \times_{p,\alpha} B) \cdot M^t_{p\beta}(\|V_{A,H}\|_\infty^{-1}, \|V_{B,H}\|_\infty^{-1})$$
$$\geq \begin{cases} V_{n-k+1}((1-t) \times_{p,\delta} A_H +_{p,\delta} t \times_{p,\delta} B_H), & \text{if } \frac{\alpha\beta}{\alpha+\beta} \geq -\frac{1}{k}, \\ V_{n-k+1}((1-t) \times^{p,\delta} A_H +^{p,\delta} t \times^{p,\delta} B_H), & \text{if } \frac{\alpha\beta}{\alpha+\beta} < -\frac{1}{k}, \end{cases}$$

where $\delta = (\alpha^{-1} + \beta^{-1} + k)^{-1}$ and $\alpha + \beta > 0$.

$L_{p,\bar{\alpha}}$-μ-surface area

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Let μ be a Borel measure on $\mathbb{R}^{n+1}$, $p \geq 1$, and $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{n+1}) \in [0,\infty]^{n+1}$. We define the $L_{p,\bar{\alpha}}$-μ-surface area of a μ-integrable set A with respect to a μ-integrable set B by

$$S_{\mu,p,\bar{\alpha}}(A,B) := \liminf_{\varepsilon\to 0^+} \frac{\mu(A +_{p,\bar{\alpha}} \varepsilon \times_{p,\bar{\alpha}} B) - \mu(A)}{\varepsilon}.$$

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★ $\mu = V_{n+1}(\cdot)$:
The L_p surface area with respect to the $L_{p,\bar{\alpha}}$-curvilinear summation has the following form,

$$S_{p,\bar{\alpha}}(A, B) = \lim_{\varepsilon \to 0^+} \frac{V_{n+1}(A +_{p,\bar{\alpha}} \varepsilon \times_{p,\bar{\alpha}} B) - V_{n+1}(A)}{\varepsilon}.$$

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★ $\bar{\alpha} = (1, \cdots, 1)$, $p \geq 1$: Wu, AAM, 2017.

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- ★ $\bar{\alpha} = (1, \cdots, 1)$, $p \geq 1$: Wu, AAM, 2017.
- ★ $\bar{\alpha} = (1, \cdots, 1)$, $p = 1$: Livshyts, ADV, 2017.

L_p-Minkowski's first inequality

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Let $p \in [1, \infty)$, $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{n+1}) \in [0, \infty]^{n+1}$ and $F \colon \mathbb{R}_+ \to \mathbb{R}$ be a differentiable invertible function. Let μ be a Borel measure on $\mathbb{R}^{n+1}$ which is $F(t)$-concave of $L_{p,\bar{\alpha}}$-curvilinear summation for any two μ-measurable sets $A, B \subset \mathbb{R}^{n+1}$, then

$$S_{\mu,p,\bar{\alpha}}(A, B) \geq S_{\mu,p,\bar{\alpha}}(A, A) + \frac{F(\mu(B)) - F(\mu(A))}{F'(\mu(A))}.$$

★ $\mu(A) = \mu(B)$: Isoperimetric inequality,

$$S_{\mu,p,\bar{\alpha}}(A, B) \geq S_{\mu,p,\bar{\alpha}}(A, A).$$

★ $p \in [1, \infty)$, $\bar{\alpha} = (\alpha_1, \cdots, , \alpha_{n+1}) \in (0, 1]^{n+1}$:

$$S_{p,\bar{\alpha}}(A, B) \geq S_{p,\bar{\alpha}}(A, A) + \frac{V_{n+1}(B)^{p\gamma} - V_{n+1}(A)^{p\gamma}}{p\gamma V_{n+1}(A)^{p\gamma - 1}},$$

where $\gamma = \left(\sum_{i=1}^{n+1} \alpha_i^{-1}\right)^{-1}$. If $V_{n+1}(A) = V_{n+1}(B) > 0$, then

$$S_{p,\bar{\alpha}}(A, B) \geq S_{p,\bar{\alpha}}(A, A).$$

Section 4

Multiple L_p Borell-Brascamp-Lieb inequality

Multiple L_p Borell-Brascamp-Lieb inequality

Let $p \geq 1$, $p^{-1} + q^{-1} = 1$, $t \in (0,1)$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$, $\alpha_i \in (0,1]$ for all $i = 1, \ldots, n$. Suppose that $f, g, h \colon (\mathbb{R}_+)^n \to \mathbb{R}_+$ are a triple of bounded integrable functions satisfying

$$h\left(M_{p,\alpha_1}^{(t,\lambda)}(x_1, y_1), \ldots, M_{p,\alpha_n}^{(t,\lambda)}(x_n, y_n)\right) \geq M_{p,\alpha_{n+1}}^{(t,\lambda)}(f(x_1, \ldots, x_n), g(y_1, \ldots, y_n))$$

for all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in (\mathbb{R}_+)^n$ such that $f(x)g(y) > 0$ and for all $\lambda \in (0,1)$.

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for all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in (\mathbb{R}_+)^n$ such that $f(x)g(y) > 0$ and for all $\lambda \in (0,1)$. Then

$$\int_{(\mathbb{R}_+)^n} h(x)dx$$
$$\geq \begin{cases} M_{p\gamma}^t\left(\int_{(\mathbb{R}_+)^n} f(x)dx, \int_{(\mathbb{R}_+)^n} g(x)dx\right), & \alpha_{n+1} \geq \beta, \\ \sup_{0<\lambda<1} \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{(\mathbb{R}_+)^n} f(x)dx, [D_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{(\mathbb{R}_+)^n} g(x)dx\right\}, & \alpha_{n+1} < \beta, \end{cases}$$

where $\beta = -\left(\sum_{i=1}^n \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_{i=1}^{n+1} \alpha_i^{-1}\right)^{-1}$.

Main idea of proof

- $\mathsf{hyp}(m) = \widetilde{\mathsf{hyp}(m)}$, $V_{n+1}(\mathsf{hyp}(m)) = \int_{\mathbb{R}^n} m(x)dx, \quad m \in \{f, g, h\}.$

Main idea of proof

- $\text{hyp}(m) = \widetilde{\text{hyp}(m)}$, $V_{n+1}(\text{hyp}(m)) = \int_{\mathbb{R}^n} m(x)dx, \quad m \in \{f, g, h\}$.
- $\bar{\alpha} = (1, \cdots, \alpha)$:

$$
\begin{aligned}
&V_{n+1}(\text{hyp}(h)) \\
&= \int_{\mathbb{R}^n} h(z)dz \\
&\geq \int_{\mathbb{R}^n} \sup_{0<\lambda<1} \left[\sup_{z = C_{p,\lambda,t}x + D_{p,\lambda,t}y} M_{p,\alpha}^{(t,\lambda)}(f(x), g(y)) \right] dz \\
&= \int_{\mathbb{R}^n} V_1([(1-t) \times_{p,\alpha} \text{hyp}(f) +_{p,\alpha} t \times_{p,\alpha} \text{hyp}(g)] \cap (\mathbb{R}_+ + z))dz \\
&= V_{n+1}((1-t) \times_{p,\alpha} \text{hyp}(f) +_{p,\alpha} t \times_{p,\alpha} \text{hyp}(g)) \\
&\geq \begin{cases} M_{p\gamma}^t(V_{n+1}(\text{hyp}(f)), V_{n+1}(\text{hyp}(g))), & \text{if } \alpha \geq -\frac{1}{n}, \\ \sup_{0<\lambda<1} \min\left\{ [C_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(\text{hyp}(f)), [D_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(\text{hyp}(g)) \right\}, & \text{if } \alpha < -\frac{1}{n} \end{cases} \\
&= \begin{cases} M_{p\gamma}^t\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right), & \text{if } \alpha \geq -\frac{1}{n}, \\ \sup_{0<\lambda<1} \left\{ \min\left\{ [C_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{\mathbb{R}^n} f, [D_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{\mathbb{R}^n} g \right\}\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}
\end{aligned}
$$

Main idea of proof

- $\text{hyp}(m) = \widetilde{\text{hyp}(m)}$, $V_{n+1}(\text{hyp}(m)) = \int_{\mathbb{R}^n} m(x)dx, \quad m \in \{f, g, h\}.$
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&= \int_{\mathbb{R}^n} V_1([(1-t)\times_{p,\alpha} \text{hyp}(f) +_{p,\alpha} t \times_{p,\alpha} \text{hyp}(g)] \cap (\mathbb{R}_+ + z))dz\\
&= V_{n+1}((1-t)\times_{p,\alpha} \text{hyp}(f) +_{p,\alpha} t\times_{p,\alpha} \text{hyp}(g))\\
&\geq \begin{cases} M_{p\gamma}^t(V_{n+1}(\text{hyp}(f)), V_{n+1}(\text{hyp}(g))), & \text{if } \alpha \geq -\frac{1}{n},\\ \sup_{0<\lambda<1} \min\left\{ [C_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(\text{hyp}(f)), [D_{p,\lambda,t}]^{\frac{1}{\gamma}} V_{n+1}(\text{hyp}(g))\right\}, & \text{if } \alpha < -\frac{1}{n} \end{cases}\\
&= \begin{cases} M_{p\gamma}^t\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right), & \text{if } \alpha \geq -\frac{1}{n},\\ \sup_{0<\lambda<1}\left\{\min\left\{ [C_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{\mathbb{R}^n} f, [D_{p,\lambda,t}]^{\frac{1}{\gamma}} \int_{\mathbb{R}^n} g\right\}\right\}, & \text{if } \alpha < -\frac{1}{n}. \end{cases}
\end{aligned}
$$

- Repeat the above with respect to the multiple power parameters.

Normalized L_p BBL inequality

Normalized L_p BBL inequality

Let $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $t \in (0,1)$, and $\alpha, \beta \in [-\infty, \infty]$ be such that $\alpha + \beta \geq 0$. Let $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of integrable functions satisfying the condition

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq M_{p,\alpha}^{(t,\lambda)}(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$ and all $\lambda \in (0,1)$.

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for all $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$ and all $\lambda \in (0,1)$.Then

$$\left(\int_{\mathbb{R}^n} h(x)dx\right) \cdot M^t_{p\beta}(\|f\|_\infty^{-1}, \|g\|_\infty^{-1})$$
$$\geq \begin{cases} M^t_{p\delta}\left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\|f\|_\infty}, \frac{\int_{\mathbb{R}^n} g(x)dx}{\|g\|_\infty}\right) & \text{if } \frac{\alpha\beta}{\alpha+\beta} \geq -\frac{1}{n}, \\ \sup_{0<\lambda<1}\left\{\min\left\{[C_{p,\lambda,t}]^{\frac{1}{\delta}} \frac{\int_{\mathbb{R}^n} f(x)dx}{\|f\|_\infty}, [D_{p,\lambda,t}]^{\frac{1}{\delta}} \frac{\int_{\mathbb{R}^n} g(x)dx}{\|g\|_\infty}\right\}\right\}, & \text{if } \frac{\alpha\beta}{\alpha+\beta} < -\frac{1}{n}, \end{cases}$$

where $\delta = (\alpha^{-1} + \beta^{-1} + n)^{-1}$.

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where $\delta = (\alpha^{-1} + \beta^{-1} + n)^{-1}$.

- We also obtained the normalized version in low-dimensional cases with $\mathbb{R}^n = H \times H^\perp$.

Multiple L_p supremal-convolution

$L_{p,\bar{\alpha}}$ supremal-convolution

We define $L_{p,\bar{\alpha}}$ supremal-convolution of $f, g : \mathbb{R}^n_+ \to \mathbb{R}_+$ as

$$((1-t)\times_{p,\bar{\alpha}} f +_{p,\bar{\alpha}} t \times_{p,\bar{\alpha}} g)(z_1, \cdots, z_n)$$

$$= \sup_{0<\lambda<1} \left(\sup_{z_i = M^{(t,\lambda)}_{p,\alpha_i}(x_i, y_i), 1 \le i \le n} M^{(t,\lambda)}_{p,\alpha_{n+1}} \left(f(x_1, \cdots, x_n), g(y_1, \cdots, y_n) \right) \right).$$

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We define $L_{p,\bar{\alpha}}$ supremal-convolution of $f, g : \mathbb{R}_+^n \to \mathbb{R}_+$ as

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Brunn-Minkowski type inequality

$$\int_{\mathbb{R}^n}((1-t)\times_{p,\bar{\alpha}} f \oplus_{p,\bar{\alpha}} t\times_{p,\bar{\alpha}} g)(x)dx$$
$$\ge \begin{cases} M_{p\gamma}^t\left(\int_{(\mathbb{R}_+)^n} f(x)dx, \int_{(\mathbb{R}_+)^n} g(x)dx\right), & \text{if } \alpha_{n+1}\ge\beta,\\ \sup_{0<\lambda<1}\min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}}\int_{(\mathbb{R}_+)^n} f(x)dx, [D_{p,\lambda,t}]^{\frac{1}{\gamma}}\int_{(\mathbb{R}_+)^n} g(x)dx\right\}, & \text{if } \alpha_{n+1}<\beta,\end{cases}$$

where $p\ge 1$, $\beta = -\left(\sum_{i=1}^n \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_{i=1}^{n+1}\alpha_i^{-1}\right)^{-1}$.

L_p-μ-surface area of a μ-integrable function

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Let μ be a Borel measure on $\mathbb{R}^n$, $p \geq 1$, and $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{n+1}) \in [0, \infty]^{n+1}$. We define the L_p-μ-surface area of a μ-integrable function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ with respect to a μ-integrable function g by

$$\mathbb{S}_{\mu,p,\bar{\alpha}}(f, g) := \liminf_{\varepsilon \to 0^+} \frac{\int_{\mathbb{R}^n} f \oplus_{p,\bar{\alpha}} (\varepsilon \times_{p,\bar{\alpha}} g) d\mu - \int_{\mathbb{R}^n} f d\mu}{\varepsilon}.$$

If μ is the Lebesgue measure on $\mathbb{R}^n$, we will simply denote it as $\mathbb{S}_{p,\bar{\alpha}}$.

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If μ is the Lebesgue measure on $\mathbb{R}^n$, we will simply denote it as $\mathbb{S}_{p,\bar{\alpha}}$.

★ $\alpha_i = 1$ for $1 \leq i \leq n$ and $\alpha_{n+1} = \frac{1}{s}$:

(1) $f \oplus_{p,\bar{\alpha}} (\varepsilon \times_{p,\bar{\alpha}} g)$ recovers the $L_{p,s}$-supremal-convolution in Roysdon-Xing, TRAMS, 2020;

(2) $\mathbb{S}_{\mu,p,\bar{\alpha}}(f,g)$ reduces to the L_p-μ-surface area of a μ-integrable function f with respect to a μ-integrable function g via $L_{p,s}$-supremal-convolution.

Minkowski's first inequality

L_p-Minkowski's first inequality in terms of $L_{p,\bar{\alpha}}$ supremal-convolution for functions

Let $p \in [1,\infty)$, $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{n+1}) \in [0,\infty]^{n+1}$, and $F\colon \mathbb{R}_+ \to \mathbb{R}$ be a differentiable invertible function. Let μ be a Borel measure on $\mathbb{R}^n$, and assume that μ is $F(t)$-concave of non-negative bounded μ-integrable functions and the $L_{p,\bar{\alpha}}$–supremal convolution. Then

$$\mathbb{S}_{\mu,p,\bar{\alpha}}(f,g) \geq \mathbb{S}_{\mu,p,\bar{\alpha}}(f,f) + \frac{F\left(\int_{\mathbb{R}^n} g d\mu\right) - F\left(\int_{\mathbb{R}^n} f d\mu\right)}{F'\left(\int_{\mathbb{R}^n} f d\mu\right)}.$$

★ $\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} g d\mu$: Isoperimetric inequality,

$$\mathbb{S}_{\mu,p,\bar{\alpha}}(f,g) \geq \mathbb{S}_{\mu,p,\bar{\alpha}}(f,f).$$

Concluding remarks

- We also obtained the normalized version of the $L_{p,\bar{\alpha}}$ curvilinear Brunn-Minkowski inequality for sets and $L_{p,\bar{\alpha}}$ Borell-Brascamp-Lieb inequality for measures for $p \geq 1$.

Concluding remarks

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  - Conjecture: Local version of $L_{p,\bar{\alpha}}$ curvilinear Brunn-Minkowski inequality for sets and $L_{p,\bar{\alpha}}$ Borell-Brascamp-Lieb inequality for functions similar to Local L_p Brunn-Minkowski inequality for convex bodies.

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  - Trial: $L_{p,\bar{\alpha}}$ curvilinear Brunn-Minkowski inequality for unconditional or symmetric sets.

Thank You !